About the photon wave function

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We construct the wave function of the photon in position space from the dispersion relation of a massless particle and the spin-one representation of the rotation group, and follow the construction through to its Lorentz covariance, its action principle and its quantization. The wave equation is first order in time, contains no Planck constant, and coincides with Maxwell's equations written for the Riemann–Silberstein vector. A photon of arbitrary polarization is described by a six-component wave function that carries the representation $(1,0)\oplus(0,1)$ of the Lorentz group and obeys an equation of Dirac type. We give the Hamiltonian, the spin and helicity operators, the plane-wave solutions, the finite Lorentz transformations in the chiral and standard representations, and the nonlocal scalar product. The squared modulus of the wave function is the energy density of the photon. We show that the sesquilinear forms of the wave function contain no four-vector, so that no first-order Lagrangian of the Dirac kind is a Lorentz scalar, and we give the two action principles that do exist, a first-order action that yields the wave equation in every frame and the Maxwell action written with the Riemann–Silberstein vector of the potentials. The conserved currents of both are obtained, and the wave equation in an inhomogeneous medium and the relation to the quantized field are derived.

I. INTRODUCTION

The two relativistic wave equations of the early quantum theory, the Klein–Gordon and Dirac equations, were written for particles with mass. The Klein–Gordon equation has no positive definite conserved density, and the Dirac equation has one at the price of solutions of negative frequency, whose interpretation leads from a theory of one particle to a theory of fields. In quantum field theory the position is a label of the field operators and not an observable of the particle [1], and the notion of a wave function in position space loses the central place it has in nonrelativistic quantum mechanics.

For the photon the difficulty is sharper. A particle of mass m cannot be localized within

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a distance smaller than its Compton wavelength $\hbar/mc$ without making the creation of further particles possible, and for $m = 0$ this length is infinite. The precise form of the statement is due to Newton and Wigner [2] and to Wightman [3]. States localized at a point exist and are unique for every massive particle, and they do not exist for a massless particle of helicity one. The photon has no position operator, and no function of position can have $|\psi(\mathbf{r}, t)|^2$ as the probability density of finding the photon at $\mathbf{r}$.

A wave function is nevertheless more than a position density. It specifies the state, it evolves by a Schrödinger equation, and it transforms in a definite way under the Poincaré group. In this sense the photon does have a wave function in position space, as shown by Bialynicki-Birula [4, 5] and by Sipe [6]. It is built from the Riemann–Silberstein vector [7, 8], its wave equation is the set of Maxwell equations, and its squared modulus is the density of the energy of the photon. The six-component form of this wave function, which transforms as the $(1, 0) \oplus (0, 1)$ representation of the Lorentz group, obeys an equation that resembles the Dirac equation [9], and the wave function has been extended to several photons and related to classical coherence theory [10]. A different proposal, in which a conserved probability current exists, is that of Kiessling and Tahvildar-Zadeh [11].

In this paper we derive the photon wave function following the route that Dirac took for the electron. We start in Sec. II from the dispersion relation of a massless particle and the matrices of spin one, and obtain a first-order wave equation and the weight that relates the wave function in momentum space to the one in position space. Section III identifies the equation with Maxwell's equations, and Sec. IV assembles the six-component wave function with its Hamiltonian, spin, helicity and plane-wave solutions. The energy density and the scalar product are obtained in Sec. V. In Sec. VI we construct the finite Lorentz transformations and prove the covariance of the wave equation. Section VII deals with the action principle, where the analogy with the Dirac theory has to be handled with care, and Sec. VIII with parity, time reversal and duality. The photon in a medium is treated in Sec. IX and the relation to the quantized field in Sec. X. We use SI units throughout.

II. WAVE EQUATION OF A MASSLESS PARTICLE OF SPIN ONE

A. Momentum space

A free massless particle of momentum $\mathbf{p}$ has the energy

$$E = c\,|\mathbf{p}|, \tag{1}$$

and we look for a Schrödinger equation $i\hbar\,\partial_t\psi = H\psi$ whose Hamiltonian reproduces this relation. Since $|\mathbf{p}| = \sqrt{\mathbf{p}\cdot\mathbf{p}}$ is not a differential operator in position space, we require, as Dirac did, a Hamiltonian linear in $\mathbf{p}$ whose square is $c^2\mathbf{p}^2$. The wave function of a particle of spin s has at least $2s+1$ components, so for $s=1$ it is a complex vector $\mathbf{f}(\mathbf{p},t)$ with three components, and the coefficients of $\mathbf{p}$ in the Hamiltonian are 3×3 matrices. The natural candidates are the generators of rotations in the spin-one representation,

$$(S_i)_{jk} = -i\,\varepsilon_{ijk}, \qquad i,j,k = 1,2,3, \tag{2}$$

that is,

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

They are Hermitian, satisfy $[S_i, S_j] = i\varepsilon_{ijk}S_k$, and have $\mathbf{S}^2 = S_1^2 + S_2^2 + S_3^2 = 2 = s(s+1)$. In this Cartesian basis the product $\mathbf{S}\cdot\mathbf{a}$ with any vector $\mathbf{a}$ is the matrix of the cross product,

$$(\mathbf{S}\cdot\mathbf{a})\,\mathbf{f} = i\,\mathbf{a}\times\mathbf{f}, \tag{4}$$

because $[(\mathbf{S}\cdot\mathbf{a})\mathbf{f}]_j = -i\varepsilon_{ijk}a_i f_k$. Applying this rule twice and using $\mathbf{a}\times(\mathbf{a}\times\mathbf{f}) = \mathbf{a}(\mathbf{a}\cdot\mathbf{f}) - \mathbf{a}^2\mathbf{f}$ we obtain

$$\left[(\mathbf{S}\cdot\mathbf{a})^2\right]_{jk} = \mathbf{a}^2\,\delta_{jk} - a_j a_k, \qquad (\mathbf{S}\cdot\mathbf{a})^3 = \mathbf{a}^2\,(\mathbf{S}\cdot\mathbf{a}), \tag{5}$$

where the second identity follows from the first because $\mathbf{S}\cdot\mathbf{a}$ annihilates $\mathbf{a}$.

By the first identity (5), the candidate Hamiltonian $c\,\mathbf{S}\cdot\mathbf{p}$ squares to

$$c^2(\mathbf{S}\cdot\mathbf{p})^2\,\mathbf{f} = c^2\left[\mathbf{p}^2\,\mathbf{f} - \mathbf{p}\,(\mathbf{p}\cdot\mathbf{f})\right]. \tag{6}$$

This is $c^2\mathbf{p}^2$ only on vectors with $\mathbf{p}\cdot\mathbf{f} = 0$. The reason is that $\mathbf{S}\cdot\mathbf{p}/|\mathbf{p}|$, the helicity, has the eigenvalues $+1$, 0 and -1, and the eigenvector of eigenvalue zero is the longitudinal vector $\mathbf{f} \parallel \mathbf{p}$. This vector has zero energy for every momentum and violates the dispersion relation (1). We therefore decompose $\mathbf{f} = \mathbf{f}_L + \mathbf{f}_T$ with $\mathbf{p}\times\mathbf{f}_L = 0$ and $\mathbf{p}\cdot\mathbf{f}_T = 0$, and admit only the transverse part as a wave function. A massless particle of spin one has two states for each momentum, not three. On transverse vectors $c\,\mathbf{S}\cdot\mathbf{p}$ has the eigenvalues $\pm c|\mathbf{p}|$, and both signs of the Hamiltonian are admissible,

$$i\hbar\,\partial_t\mathbf{f}_\pm = \pm c\,(\mathbf{S}\cdot\mathbf{p})\,\mathbf{f}_\pm, \qquad \mathbf{p}\cdot\mathbf{f}_\pm = 0. \tag{7}$$

The solutions of positive energy of the equation with the upper sign have helicity +1, and those of the equation with the lower sign have helicity −1. Equations (7) are the spin-one counterparts of the two Weyl equations $i\hbar\partial_t\phi = \pm c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\phi$, with the transversality condition as the new element.

For each wave vector $\mathbf{k} = \mathbf{p}/\hbar$ we introduce the circular polarization vectors $\mathbf{e}_\lambda(\mathbf{k})$, $\lambda = \pm 1$, defined by

$$(\mathbf{S}\cdot\hat{\mathbf{k}})\,\mathbf{e}_\lambda = i\,\hat{\mathbf{k}}\times\mathbf{e}_\lambda = \lambda\,\mathbf{e}_\lambda, \qquad \hat{\mathbf{k}}\cdot\mathbf{e}_\lambda = 0, \qquad \mathbf{e}_\lambda^*\cdot\mathbf{e}_{\lambda'} = \delta_{\lambda\lambda'}, \qquad \mathbf{e}_{-\lambda} = \mathbf{e}_\lambda^*, \tag{8}$$

with $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$. For $\mathbf{k}$ along the z axis they are $(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})/\sqrt{2}$. A state of positive energy is specified by two amplitudes $f_\lambda(\mathbf{k})$,

$$\mathbf{f}_\lambda(\mathbf{k},t) = \mathbf{e}_\lambda(\mathbf{k})\,f_\lambda(\mathbf{k})\,e^{-i\omega_k t}, \qquad \omega_k = c|\mathbf{k}|, \tag{9}$$

and since momentum and helicity are observables of the photon about which there is no doubt, these amplitudes have the usual meaning: $|f_\lambda(\mathbf{k})|^2\mathrm{d}^3k$ is the probability of finding the helicity λ and the wave vector in d^3k, with

$$\sum_{\lambda=\pm1}\int \mathrm{d}^3k\;|f_\lambda(\mathbf{k})|^2 = 1. \tag{10}$$

B. Position space and the weight function

The wave function in position space is a Fourier transform of $\mathbf{f}_\lambda$, and the relativistic theory leaves a freedom that is absent in the nonrelativistic one. Any weight $w(\omega_k)$ may be inserted,

$$\boldsymbol{\psi}_\lambda(\mathbf{r},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\;w(\omega_k)\,\mathbf{e}_\lambda(\mathbf{k})\,f_\lambda(\mathbf{k})\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}, \tag{11}$$

without changing the wave equation, because w depends on $\mathbf{k}$ only through $|\mathbf{k}|$ and commutes with the Hamiltonian. With $\mathbf{p} = -i\hbar\nabla$ and the rule (4), the operator $\mathbf{S}\cdot\mathbf{p}$ is $\hbar$ times the curl, and Eqs. (7) become

$$i\,\partial_t\boldsymbol{\psi}_\pm(\mathbf{r},t) = \pm c\,\nabla\times\boldsymbol{\psi}_\pm(\mathbf{r},t), \qquad \nabla\cdot\boldsymbol{\psi}_\pm(\mathbf{r},t) = 0. \tag{12}$$

Planck's constant has dropped out. This happens only for zero mass, since a mass would enter the equation through $mc^2/\hbar$.

The weight is fixed by what we require of $|\boldsymbol{\psi}|^2$. The choice $w = 1$ would make $\int \mathrm{d}^3r\,\sum_\lambda|\boldsymbol{\psi}_\lambda|^2 = 1$, but the function so defined has no simple transformation law under Lorentz boosts and no local relation to any field, and by the theorem of Newton and Wigner its squared modulus cannot be

a position probability density in any case. The Hamiltonian is one of the few observables of the photon that is local in position space, and we require instead that the integral of $|\boldsymbol{\psi}|^2$ be the mean energy. From Eq. (11),

$$\sum_\lambda \int \mathrm{d}^3 r \; |\boldsymbol{\psi}_\lambda(\mathbf{r},t)|^2 = \sum_\lambda \int \mathrm{d}^3 k \; w(\omega_k)^2 \, |f_\lambda(\mathbf{k})|^2, \tag{13}$$

which equals $\langle H \rangle = \sum_\lambda \int \mathrm{d}^3 k \, \hbar\omega_k |f_\lambda|^2$ for

$$w(\omega_k) = \sqrt{\hbar\omega_k}. \tag{14}$$

With this weight the wave function is

$$\boldsymbol{\psi}_\lambda(\mathbf{r},t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \sqrt{\hbar\omega_k} \; \mathbf{e}_\lambda(\mathbf{k}) \, f_\lambda(\mathbf{k}) \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)}, \qquad \lambda = \pm 1. \tag{15}$$

We show in Sec. VI that the same weight gives $\boldsymbol{\psi}_\lambda$ a local transformation law under the Lorentz group, and in Sec. X that it is the weight with which the quantized field creates a photon.

III. MAXWELL'S EQUATIONS

In empty space the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ obey

$$\partial_t \mathbf{E} = c^2 \, \nabla \times \mathbf{B}, \qquad \partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \qquad \nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0, \tag{16}$$

with $c = 1/\sqrt{\varepsilon_0 \mu_0}$. The Riemann–Silberstein vector [7, 8] is the complex combination

$$\mathbf{F} = \frac{1}{\sqrt{2}} \left(\sqrt{\varepsilon_0} \, \mathbf{E} + \frac{i}{\sqrt{\mu_0}} \, \mathbf{B} \right), \tag{17}$$

normalized so that

$$\mathbf{F}^* \cdot \mathbf{F} = \frac{\varepsilon_0}{2} \, \mathbf{E}^2 + \frac{1}{2\mu_0} \, \mathbf{B}^2 \tag{18}$$

is the energy density of the field. Maxwell's equations (16) are equivalent to

$$i \, \partial_t \mathbf{F} = c \, \nabla \times \mathbf{F}, \qquad \nabla \cdot \mathbf{F} = 0, \tag{19}$$

and $\mathbf{F}^*$ obeys the same equations with $c \to -c$. These are the wave equations (12). The equation that Maxwell wrote in 1864 for the classical field is the wave equation that the dispersion relation and the spin dictate for a massless particle of spin one, and the transversality condition is the pair of Maxwell equations without time derivatives.

The classical field $\mathbf{F}$ is not itself a wave function, because $\mathbf{E}$ and $\mathbf{B}$ are real and $\mathbf{F}$ then contains both signs of the frequency. Inserting a plane wave $\mathbf{e}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ in Eq. (19) gives $\omega\,\mathbf{e} = c\,(\mathbf{S}\cdot\mathbf{k})\,\mathbf{e}$, so that $\omega = +c|\mathbf{k}|$ goes with helicity $+1$ and $\omega = -c|\mathbf{k}|$ with helicity -1. The general solution is

$$\mathbf{F}(\mathbf{r},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\,\sqrt{\hbar\omega_k}\;\mathbf{e}_+(\mathbf{k})\left[\alpha_+(\mathbf{k})\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + \alpha_-^*(\mathbf{k})\,e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right], \tag{20}$$

with two independent complex amplitudes. The second term is the complex conjugate of a wave of positive frequency with polarization $\mathbf{e}_+^* = \mathbf{e}_-$, that is, of a wave of helicity -1. A classical field thus stores its two circular polarizations in the two signs of the frequency of $\mathbf{F}$. A quantum state has only positive frequencies, and the two polarizations must then be stored in two vectors, the positive-frequency part of $\mathbf{F}$ and the positive-frequency part of $\mathbf{F}^*$. These are the functions $\boldsymbol{\psi}_+$ and $\boldsymbol{\psi}_-$ of Eq. (15), with f_λ in the place of α_λ.

IV. THE SIX-COMPONENT WAVE FUNCTION

A. Hamiltonian

A photon of arbitrary polarization is described by the pair

$$\Psi(\mathbf{r},t) = \begin{pmatrix} \boldsymbol{\psi}_+(\mathbf{r},t) \\ \boldsymbol{\psi}_-(\mathbf{r},t) \end{pmatrix}, \tag{21}$$

and Eqs. (12) combine into a Schrödinger equation,

$$i\hbar\,\partial_t\Psi = H\Psi, \qquad H = c\begin{pmatrix} \mathbf{S}\cdot\mathbf{p} & 0 \\ 0 & -\mathbf{S}\cdot\mathbf{p} \end{pmatrix} = c\,\rho_3\,(\mathbf{S}\cdot\mathbf{p}), \qquad \nabla\cdot\boldsymbol{\psi}_\pm = 0, \tag{22}$$

where ρ_1, ρ_2, ρ_3 denote the Pauli matrices acting on the two halves of Ψ, each entry being a 3×3 block. The Hamiltonian is Hermitian because ρ_3, S_i and p_i are Hermitian and commute with each other. Its square is, by Eq. (6),

$$H^2\,\Psi = c^2\begin{pmatrix} \mathbf{p}^2 - \mathbf{p}\,\mathbf{p}^T & 0 \\ 0 & \mathbf{p}^2 - \mathbf{p}\,\mathbf{p}^T \end{pmatrix}\Psi = c^2\mathbf{p}^2\,\Psi, \tag{23}$$

where the last step uses the transversality condition. Every component of Ψ therefore obeys the wave equation

$$\left(\partial_t^2 - c^2\nabla^2\right)\Psi = 0, \tag{24}$$

which is the Klein–Gordon equation for zero mass. The converse does not hold. The second-order equation admits both signs of the energy for each helicity, while the first-order equation ties the sign of the energy to the helicity in each half of Ψ.

B. Spin and helicity

Rotations act on $\boldsymbol{\psi}_+$ and $\boldsymbol{\psi}_-$ as on vectors, so the spin operator is

$$\boldsymbol{\Sigma} = \begin{pmatrix} \mathbf{S} & 0 \\ 0 & \mathbf{S} \end{pmatrix}, \qquad \boldsymbol{\Sigma}^2 = 2, \tag{25}$$

and the wave function has spin one. The total angular momentum $\mathbf{J} = \mathbf{r} \times \mathbf{p} + \boldsymbol{\Sigma}$ commutes with H, while the orbital and spin parts do not commute with it separately. The helicity operator is the projection of the spin on the direction of the momentum,

$$\Lambda = \frac{\boldsymbol{\Sigma} \cdot \mathbf{p}}{|\mathbf{p}|}, \qquad H = c\,|\mathbf{p}|\,\rho_3\,\Lambda. \tag{26}$$

It commutes with H and has the eigenvalues $+1$, 0 and -1, each twice. The eigenvalue 0 belongs to the longitudinal vectors, which the transversality condition excludes. Of the four remaining combinations of $\rho_3 = \pm 1$ and $\Lambda = \pm 1$, two have positive energy, $\rho_3 = \Lambda = +1$ and $\rho_3 = \Lambda = -1$, and two have negative energy. The physical states are those of positive energy, on which

$$\Lambda\,\Psi = \rho_3\,\Psi, \qquad H\,\Psi = c\,|\mathbf{p}|\,\Psi. \tag{27}$$

The upper half of Ψ carries helicity $+1$ and the lower half helicity -1. The solutions of negative energy are not antiparticle states. The classical field (20) shows what they are: the negative-frequency solutions of the upper equation are the complex conjugates of the positive-frequency solutions of the lower one, and they duplicate the information already contained in $\boldsymbol{\psi}_-$. This is the form that the statement "the photon is its own antiparticle" takes for the wave function.

C. Plane waves

To exhibit the spectrum explicitly, we insert $\Psi = (\mathbf{b}_1, \mathbf{b}_2)^T e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$, with constant vectors $\mathbf{b}_1$ and $\mathbf{b}_2$, in the wave equation (22) without the constraint. The condition for a solution is

$$\det \begin{pmatrix} \omega - c\,\mathbf{S}\cdot\mathbf{k} & 0 \\ 0 & \omega + c\,\mathbf{S}\cdot\mathbf{k} \end{pmatrix} = \omega^2 \left(\omega^2 - c^2\mathbf{k}^2\right)^2 = 0. \tag{28}$$

The root $\omega = 0$ is double and belongs to $\mathbf{b}_{1,2} \parallel \mathbf{k}$, the static longitudinal fields that the constraint removes. The roots $\omega = \pm c|\mathbf{k}|$ are each double. For $\omega = +c|\mathbf{k}|$ the solutions are $\mathbf{b}_1 = \mathbf{e}_+(\mathbf{k})$, $\mathbf{b}_2 = 0$ and $\mathbf{b}_1 = 0$, $\mathbf{b}_2 = \mathbf{e}_-(\mathbf{k})$, since $(\mathbf{S}\cdot\hat{\mathbf{k}})\mathbf{e}_\pm = \pm\mathbf{e}_\pm$. The general solution of positive energy is the superposition

$$\Psi(\mathbf{r}, t) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \sqrt{\hbar\omega_k} \begin{pmatrix} \mathbf{e}_+(\mathbf{k})\, f_+(\mathbf{k}) \\ \mathbf{e}_-(\mathbf{k})\, f_-(\mathbf{k}) \end{pmatrix} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)}, \tag{29}$$

which is Eq. (15).

V. ENERGY DENSITY AND SCALAR PRODUCT

With the weight (14), the integral of $\Psi^\dagger\Psi = |\boldsymbol{\psi}_+|^2 + |\boldsymbol{\psi}_-|^2$ is the mean energy of the photon,

$$\int \mathrm{d}^3r\ \Psi^\dagger(\mathbf{r},t)\,\Psi(\mathbf{r},t) = \sum_\lambda \int \mathrm{d}^3k\ \hbar\omega_k\,|f_\lambda(\mathbf{k})|^2 = \langle H\rangle, \tag{30}$$

in agreement with the classical meaning (18) of $\mathbf{F}^*\cdot\mathbf{F}$. The density is locally conserved. From the wave equation and $\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\nabla\times\mathbf{a} - \mathbf{a}\cdot\nabla\times\mathbf{b}$,

$$\partial_t\Big(\Psi^\dagger\Psi\Big) + \nabla\cdot\mathbf{j} = 0, \qquad \mathbf{j} = c\,\Psi^\dagger\rho_3\,\boldsymbol{\Sigma}\,\Psi = -ic\left(\boldsymbol{\psi}_+^*\times\boldsymbol{\psi}_+ - \boldsymbol{\psi}_-^*\times\boldsymbol{\psi}_-\right). \tag{31}$$

For a classical field, with $\boldsymbol{\psi}_+ \to \mathbf{F}$ and no lower component, $\mathbf{j}$ is the Poynting vector $\mathbf{E}\times\mathbf{B}/\mu_0$ and Eq. (31) is Poynting's theorem. For the photon the integral of the flux is the mean momentum,

$$\frac{1}{c^2}\int \mathrm{d}^3r\ \mathbf{j}(\mathbf{r},t) = \sum_\lambda \int \mathrm{d}^3k\ \hbar\mathbf{k}\,|f_\lambda(\mathbf{k})|^2 = \langle\mathbf{p}\rangle, \tag{32}$$

as follows from $-i\,\mathbf{e}_\lambda^*\times\mathbf{e}_\lambda = \lambda\,\hat{\mathbf{k}}$. The ratio $\Psi^\dagger\Psi/\langle H\rangle$ is positive, normalized and locally conserved, and it is the fraction of the energy of the photon found near $\mathbf{r}$ [4–6].

The probability is given by the momentum-space amplitudes, and the scalar product of two states with amplitudes f_λ and g_λ is

$$\langle f|g\rangle = \sum_\lambda \int \mathrm{d}^3k\ f_\lambda^*(\mathbf{k})\,g_\lambda(\mathbf{k}). \tag{33}$$

The energy cannot serve as a scalar product for transition amplitudes, because the probability must be a Lorentz scalar and the energy is not. To express Eq. (33) through the wave functions we invert the Fourier transform (15) at a fixed time, $\sqrt{\hbar\omega_k}\,\mathbf{e}_\lambda f_\lambda e^{-i\omega_k t} = (2\pi)^{-3/2}\int \mathrm{d}^3r\,\boldsymbol{\psi}_\lambda(\mathbf{r},t)e^{-i\mathbf{k}\cdot\mathbf{r}}$, and find

$$\langle f|g\rangle = \int \mathrm{d}^3r \int \mathrm{d}^3r'\ \Psi_f^\dagger(\mathbf{r},t)\,\Psi_g(\mathbf{r}',t)\ K(\mathbf{r}-\mathbf{r}'), \qquad K(\mathbf{R}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{e^{i\mathbf{k}\cdot\mathbf{R}}}{\hbar c\,|\mathbf{k}|}. \tag{34}$$

The angular integration gives $4\pi\sin(kR)/kR$ with $R = |\mathbf{R}|$, so that

$$K(\mathbf{R}) = \frac{1}{2\pi^2\hbar c\,R}\int_0^\infty \mathrm{d}k\ \sin(kR). \tag{35}$$

The remaining integral does not converge in the ordinary sense. As a distribution it is defined with a convergence factor $e^{-\epsilon k}$,

$$\int_0^\infty \mathrm{d}k\ \sin(kR)\,e^{-\epsilon k} = \frac{R}{R^2+\epsilon^2} \ \xrightarrow[\epsilon\to 0]{}\ \frac{1}{R}, \tag{36}$$

and the scalar product is

$$\langle f|g\rangle = \frac{1}{2\pi^2\hbar c}\int \mathrm{d}^3r \int \mathrm{d}^3r' \; \frac{\Psi_f^\dagger(\mathbf{r},t)\,\Psi_g(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|^2}. \tag{37}$$

Apart from normalization conventions this is the scalar product of Ref. [5], where its invariance under the Poincaré group is shown. The norm is a double integral and cannot be written as the integral of a local density, which is how the absence of a position operator appears in this formalism. The Hamiltonian is Hermitian with respect to the scalar product (37), because it commutes with the multiplication by $1/|\mathbf{p}|$, and $\langle f|H|f\rangle$ is the local integral (30).

VI. LORENTZ COVARIANCE

A. The representations $(1,0)$ and $(0,1)$

Let $\mathbf{J}$ and $\mathbf{K}$ be the generators of rotations and boosts, so that a rotation by the vector angle $\boldsymbol{\theta}$ is represented by $e^{-i\boldsymbol{\theta}\cdot\mathbf{J}}$ and a boost of rapidity $\boldsymbol{\eta}$ by $e^{-i\boldsymbol{\eta}\cdot\mathbf{K}}$. They satisfy

$$[J_i, J_j] = i\varepsilon_{ijk}J_k, \qquad [J_i, K_j] = i\varepsilon_{ijk}K_k, \qquad [K_i, K_j] = -i\varepsilon_{ijk}J_k. \tag{38}$$

The combinations $\mathbf{J}^\pm = (\mathbf{J} \pm i\mathbf{K})/2$ obey

$$[J_i^\pm, J_j^\pm] = i\varepsilon_{ijk}J_k^\pm, \qquad [J_i^+, J_j^-] = 0, \tag{39}$$

so the Lie algebra of the Lorentz group is the sum of two commuting copies of the algebra of rotations, and its finite-dimensional irreducible representations are labelled by two spins (j^+, j^-) and have dimension $(2j^+ + 1)(2j^- + 1)$ [1]. The Weyl spinors are $(1/2, 0)$ and $(0, 1/2)$. A three-component object of spin one is obtained from $(1,0)$, with $\mathbf{J}^+ = \mathbf{S}$ and $\mathbf{J}^- = 0$, or from $(0,1)$, with $\mathbf{J}^+ = 0$ and $\mathbf{J}^- = \mathbf{S}$. Since $\mathbf{J} = \mathbf{J}^+ + \mathbf{J}^-$ and $\mathbf{K} = -i(\mathbf{J}^+ - \mathbf{J}^-)$,

$$(1,0): \quad \mathbf{J} = \mathbf{S}, \quad \mathbf{K} = -i\mathbf{S}; \qquad\qquad (0,1): \quad \mathbf{J} = \mathbf{S}, \quad \mathbf{K} = +i\mathbf{S}. \tag{40}$$

The finite transformations are

$$L = e^{-i\boldsymbol{\theta}\cdot\mathbf{S} - \boldsymbol{\eta}\cdot\mathbf{S}} \quad \text{on } (1,0), \qquad L^* = e^{-i\boldsymbol{\theta}\cdot\mathbf{S} + \boldsymbol{\eta}\cdot\mathbf{S}} \quad \text{on } (0,1), \tag{41}$$

which are complex conjugates of each other because the matrices S_i are imaginary. The exponent is an antisymmetric matrix, so L is a complex orthogonal matrix, $L^T L = 1$. It is unitary only for rotations.

The exponential can be summed with the second identity (5). For a unit vector $\mathbf{n}$ and any complex number z,

$$e^{-iz\,\mathbf{n}\cdot\mathbf{S}} = 1 - i\sin z\;(\mathbf{n}\cdot\mathbf{S}) - (1-\cos z)\,(\mathbf{n}\cdot\mathbf{S})^2. \tag{42}$$

For $z=\theta$ real this is the rotation by θ about $\mathbf{n}$. A boost of rapidity η along $\mathbf{n}$ is the same formula at $z=-i\eta$, a rotation by an imaginary angle, and acting on a vector $\mathbf{F}$ it gives

$$L\,\mathbf{F} = \cosh\eta\;\mathbf{F} - (\cosh\eta - 1)\,\mathbf{n}\,(\mathbf{n}\cdot\mathbf{F}) - i\sinh\eta\;\mathbf{n}\times\mathbf{F}. \tag{43}$$

That the Riemann–Silberstein vector is the $(1,0)$ object follows by comparison with the transformation of the fields. For a frame moving with velocity $v = c\tanh\eta$ along x [12],

$$\begin{aligned} E'_x &= E_x, \quad E'_y = \gamma\,(E_y - vB_z), \qquad E'_z = \gamma\,(E_z + vB_y), \\ B'_x &= B_x, \quad B'_y = \gamma\,(B_y + vE_z/c^2), \quad B'_z = \gamma\,(B_z - vE_y/c^2), \end{aligned} \tag{44}$$

with $\gamma = \cosh\eta$, and the combination (17) of these is

$$F'_x = F_x, \qquad F'_y = \cosh\eta\,F_y + i\sinh\eta\,F_z, \qquad F'_z = \cosh\eta\,F_z - i\sinh\eta\,F_y, \tag{45}$$

which is Eq. (43) for $\mathbf{n}=\hat{\mathbf{x}}$. Hence $\mathbf{F}' = L\mathbf{F}$ and $\mathbf{F}^{*\prime} = L^*\mathbf{F}^*$, the fields on both sides being taken at the same event. The orthogonality of L gives the invariant

$$\mathbf{F}\cdot\mathbf{F} = \frac{1}{2}\left(\varepsilon_0\,\mathbf{E}^2 - \frac{\mathbf{B}^2}{\mu_0}\right) + i\sqrt{\frac{\varepsilon_0}{\mu_0}}\;\mathbf{E}\cdot\mathbf{B}, \tag{46}$$

whose real and imaginary parts are the two invariants of the electromagnetic field.

The positive-frequency parts of $\mathbf{F}$ and $\mathbf{F}^*$ transform with the same matrices, since the sign of the frequency is Lorentz invariant. The photon wave function therefore transforms as

$$\Psi'(x') = T\,\Psi(x), \qquad T = \begin{pmatrix} L & 0 \\ 0 & L^* \end{pmatrix} = \exp[-i\boldsymbol{\theta}\cdot\boldsymbol{\Sigma} - \boldsymbol{\eta}\cdot\rho_3\boldsymbol{\Sigma}]\,, \tag{47}$$

the representation $(1,0)\oplus(0,1)$, with generators $\mathbf{J}=\boldsymbol{\Sigma}$ and $\mathbf{K} = -i\rho_3\boldsymbol{\Sigma}$. In covariant notation, with $\theta_{ij} = \varepsilon_{ijk}\theta_k$, $\theta_{0i} = -\theta_{i0} = \eta_i$ and

$$G^{ij} = \varepsilon_{ijk}\,\Sigma_k, \qquad G^{0i} = -G^{i0} = -i\rho_3\,\Sigma_i, \tag{48}$$

the transformation is $T = \exp(-\frac{i}{2}\theta_{\mu\nu}G^{\mu\nu})$. This local law is a consequence of the weight (14). With any other weight the transformed wave function at a point would depend on the original one in a neighbourhood of that point.

B. Standard representation and the equation of Dirac type

The basis (21), in which the Lorentz transformations are block diagonal, is the analogue of the chiral representation of the Dirac theory. The analogue of the standard representation is obtained with the unitary and Hermitian matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \Psi_D = U\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \boldsymbol{\psi}_+ + \boldsymbol{\psi}_- \\ \boldsymbol{\psi}_+ - \boldsymbol{\psi}_- \end{pmatrix}. \tag{49}$$

For a classical field, where the two halves are $\mathbf{F}$ and $\mathbf{F}^*$, this is

$$\chi = U \begin{pmatrix} \mathbf{F} \\ \mathbf{F}^* \end{pmatrix} = \begin{pmatrix} \sqrt{\varepsilon_0}\,\mathbf{E} \\ i\,\mathbf{B}/\sqrt{\mu_0} \end{pmatrix}, \tag{50}$$

with the electric field in the upper half and the magnetic field in the lower half. Under U the matrices transform as $\rho_3 \to \rho_1$ and $\rho_1 \to \rho_3$, while $\boldsymbol{\Sigma}$ is unchanged, so that

$$i\hbar\,\partial_t \Psi_D = H_D \Psi_D, \qquad H_D = c\,\boldsymbol{\alpha}\cdot\mathbf{p}, \qquad \boldsymbol{\alpha} = \rho_1 \boldsymbol{\Sigma} = \begin{pmatrix} 0 & \mathbf{S} \\ \mathbf{S} & 0 \end{pmatrix}. \tag{51}$$

Multiplying by $\beta^0 = \rho_3$ and writing $x^0 = ct$, we obtain an equation of Dirac form [9],

$$\beta^\mu \partial_\mu \Psi_D = 0, \qquad \beta^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \beta^i = \beta^0 \alpha_i = \begin{pmatrix} 0 & S_i \\ -S_i & 0 \end{pmatrix}. \tag{52}$$

The analogy has limits that matter in what follows. The matrices β^μ do not satisfy the Clifford algebra, since by Eq. (6) $(\boldsymbol{\alpha}\cdot\mathbf{p})^2 = \mathbf{p}^2$ holds only on transverse wave functions, and the constraint has to be carried along with the equation.

The Lorentz transformation in this representation is

$$T_D = UTU = \frac{1}{2} \begin{pmatrix} L + L^* & L - L^* \\ L - L^* & L + L^* \end{pmatrix} = \exp[-i\boldsymbol{\theta}\cdot\boldsymbol{\Sigma} - \boldsymbol{\eta}\cdot\boldsymbol{\alpha}]\,. \tag{53}$$

Its inverse follows from the orthogonality of L. In the chiral representation

$$\rho_1 T^\dagger \rho_1 = \begin{pmatrix} L^T & 0 \\ 0 & L^{*T} \end{pmatrix} = T^{-1}, \tag{54}$$

and since $U\rho_1 U = \rho_3 = \beta^0$,

$$T_D^{-1} = \beta^0\, T_D^\dagger\, \beta^0. \tag{55}$$

With the adjoint $\bar\Psi_D = \Psi_D^\dagger\beta^0$ it follows that $\bar\Psi_D\Psi_D$ is a Lorentz scalar. For the classical field $\bar\chi\chi = \varepsilon_0\mathbf{E}^2 - \mathbf{B}^2/\mu_0$, twice the real part of the invariant (46), while $\chi^\dagger\chi$ is twice the energy density and is not invariant. The projectors on the two irreducible parts are

$$P_{(1,0)} = \frac{1+\rho_3}{2}, \qquad P_{(0,1)} = \frac{1-\rho_3}{2} \tag{56}$$

in the chiral representation, and $(1\pm\rho_1)/2$ in the standard one.

For a boost along x, Eq. (45) gives

$$\begin{gathered} L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\eta & i\sinh\eta \\ 0 & -i\sinh\eta & \cosh\eta \end{pmatrix}, \\ \frac{L+L^*}{2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\eta & 0 \\ 0 & 0 & \cosh\eta \end{pmatrix}, \qquad \frac{L-L^*}{2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i\sinh\eta \\ 0 & -i\sinh\eta & 0 \end{pmatrix}, \end{gathered} \tag{57}$$

and $\chi' = T_D\chi$ reads, in components,

$$\sqrt{\varepsilon_0}\,\mathbf{E}' = \begin{pmatrix} \sqrt{\varepsilon_0}\,E_x \\ \cosh\eta\,\sqrt{\varepsilon_0}\,E_y - \sinh\eta\,B_z/\sqrt{\mu_0} \\ \cosh\eta\,\sqrt{\varepsilon_0}\,E_z + \sinh\eta\,B_y/\sqrt{\mu_0} \end{pmatrix}, \qquad \frac{\mathbf{B}'}{\sqrt{\mu_0}} = \begin{pmatrix} B_x/\sqrt{\mu_0} \\ \cosh\eta\,B_y/\sqrt{\mu_0} + \sinh\eta\,\sqrt{\varepsilon_0}\,E_z \\ \cosh\eta\,B_z/\sqrt{\mu_0} - \sinh\eta\,\sqrt{\varepsilon_0}\,E_y \end{pmatrix}. \tag{58}$$

With $\gamma = \cosh\eta$, $\gamma v/c = \sinh\eta$ and $c = 1/\sqrt{\varepsilon_0\mu_0}$ these are the transformation rules (44), which checks the construction.

C. Covariance of the wave equation

It remains to show that the wave equation and the constraint keep their form in every frame. Neither does so alone. Consider a function $\mathbf{f}(x)$ that transforms as $(1,0)$, and form the four quantities

$$\mathcal{J}^0 = c\,\nabla\cdot\mathbf{f}, \qquad \boldsymbol{\mathcal{J}} = -\partial_t\mathbf{f} - ic\,\nabla\times\mathbf{f}, \tag{59}$$

which vanish when $\mathbf{f}$ obeys Eqs. (12) with the upper sign. Under an infinitesimal boost with $\mathbf{u} = \mathbf{v}/c$, the function changes as $\mathbf{f}' = \mathbf{f} - i\,\mathbf{u}\times\mathbf{f}$ by Eq. (43), and the derivatives as $\partial_t' = \partial_t + c\,\mathbf{u}\cdot\nabla$ and $\nabla' = \nabla + \mathbf{u}\,\partial_t/c$. Then, to first order in $\mathbf{u}$,

$$\begin{aligned} c\,\nabla'\cdot\mathbf{f}' &= c\,\nabla\cdot\mathbf{f} + \mathbf{u}\cdot\partial_t\mathbf{f} + ic\,\mathbf{u}\cdot\nabla\times\mathbf{f} = \mathcal{J}^0 - \mathbf{u}\cdot\boldsymbol{\mathcal{J}}, \\ -\partial_t'\mathbf{f}' - ic\,\nabla'\times\mathbf{f}' &= -\partial_t\mathbf{f} - ic\,\nabla\times\mathbf{f} - c\,\mathbf{u}\,(\nabla\cdot\mathbf{f}) = \boldsymbol{\mathcal{J}} - \mathbf{u}\,\mathcal{J}^0, \end{aligned} \tag{60}$$

where we used $\nabla \times (\mathbf{u} \times \mathbf{f}) = \mathbf{u}\,(\nabla \cdot \mathbf{f}) - (\mathbf{u} \cdot \nabla)\mathbf{f}$ and the cancellation of the terms $\mathbf{u} \times \partial_t \mathbf{f}$ and $(\mathbf{u} \cdot \nabla)\mathbf{f}$. Equations (60) are the infinitesimal boost of a four-vector, and under rotations $\mathcal{J}^0$ is a scalar and $\boldsymbol{\mathcal{J}}$ a vector. The four quantities (59) therefore form a four-vector $\mathcal{J}^\mu$, and the complete set of equations is the covariant statement

$$\mathcal{J}^\mu = 0. \tag{61}$$

A boost mixes the wave equation with the transversality condition, and the two together are covariant. With sources the same four-vector is proportional to the electric four-current, $\mathcal{J}^\mu = (c\rho, \mathbf{J})/\sqrt{2\varepsilon_0}$. The equations for the $(0,1)$ half follow by complex conjugation, and the covariance of Eq. (22) under the transformation (47) is established.

VII. ACTION PRINCIPLE

A. Lorentz structure of the bilinears

The form (52) suggests the Lagrangian of the Dirac theory, $\bar{\Psi}_D \beta^\mu \partial_\mu \Psi_D$, and its conserved current $\bar{\Psi}_D \beta^\mu \Psi_D$. In the Dirac theory $\bar{\psi}\gamma^\mu\psi$ is a four-vector because the product of the representation of ψ with its conjugate contains $(1/2, 1/2)$. For the photon the conjugate of a $(1,0)$ object is $(0,1)$, and the sesquilinear forms of $\Psi = (\boldsymbol{\psi}_+, \boldsymbol{\psi}_-)$ decompose as

$$\boldsymbol{\psi}_\pm^* \otimes \boldsymbol{\psi}_\pm : \ (0,1) \otimes (1,0) = (1,1), \qquad \boldsymbol{\psi}_+^* \otimes \boldsymbol{\psi}_- : \ (0,1) \otimes (0,1) = (0,0) \oplus (0,1) \oplus (0,2), \tag{62}$$

together with the conjugate of the second product. None of these contains $(1/2, 1/2)$, so no sesquilinear form of the photon wave function is a four-vector. The diagonal products give the symmetric traceless tensor $(1,1)$, whose components are the energy density $\Psi^\dagger\Psi$, the flux $\mathbf{j}$ of Eq. (31) and the stress tensor. The four quantities $\bar{\Psi}_D \beta^\mu \Psi_D$ belong to it, since $\bar{\Psi}_D \beta^0 \Psi_D = \Psi_D^\dagger \Psi_D$ and $c\,\bar{\Psi}_D \beta^i \Psi_D = j_i$. The mixed products give the scalar $\boldsymbol{\psi}_+^* \cdot \boldsymbol{\psi}_-$, which is invariant because L is orthogonal, and whose real part is $\frac{1}{2}\bar{\Psi}_D \Psi_D$. A first-order Lagrangian contains one derivative, which transforms as $(1/2, 1/2)$, and

$$(1,1) \otimes (\tfrac{1}{2}, \tfrac{1}{2}) \not\ni (0,0), \qquad [(0,0) \oplus (0,1) \oplus (0,2)] \otimes (\tfrac{1}{2}, \tfrac{1}{2}) \not\ni (0,0). \tag{63}$$

No first-order Lagrangian sesquilinear in Ψ is a Lorentz scalar, and in particular $\bar{\Psi}_D \beta^\mu \partial_\mu \Psi_D$ is not. The same holds for bilinear forms, since $(1,0) \otimes (1,0) = (0,0) \oplus (1,0) \oplus (2,0)$ contains no $(1/2, 1/2)$ either. The invariant bilinear without derivatives is $\mathbf{F} \cdot \mathbf{F}$ of Eq. (46). An invariant action

must therefore either contain two derivatives of the wave function or use other variables, and we construct one of the second kind below.

A second point concerns the classical field. If the wave function is replaced by the real fields, as in Eq. (50), then $\bar\chi = \chi^\dagger\beta^0 = \chi^T$ is not independent of χ, the matrices β^μ are symmetric, and

$$\bar\chi\,\beta^\mu\partial_\mu\chi = \frac{1}{2}\,\partial_\mu(\chi^T\beta^\mu\chi) = \frac{1}{c}\left[\partial_t\left(\frac{\varepsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2\right) + \nabla\cdot\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}\right] \tag{64}$$

identically. For real fields the expression is a total derivative and has no Euler–Lagrange equations. A first-order action exists only for the complex wave function, whose two halves are independent of their conjugates.

B. First-order action

The action of a Schrödinger equation is $\int \mathrm{d}t\,\langle\Psi|(i\hbar\partial_t - H)|\Psi\rangle$. With the scalar product (33) it reads

$$S_0 = \sum_\lambda \int \mathrm{d}t \int \mathrm{d}^3k\; f^*_\lambda(\mathbf{k},t)\,(i\hbar\,\partial_t - \hbar\omega_k)\,f_\lambda(\mathbf{k},t), \tag{65}$$

where the amplitudes are now arbitrary functions of time and are varied independently of their conjugates. In position space S_0 is nonlocal, with the kernel of Eq. (37). Because the equation is linear and the weight commutes with H, the same equations follow from the local action obtained by inserting one power of $\hbar\omega_k$,

$$S_1 = \int \mathrm{d}t \int \mathrm{d}^3r\;\mathcal{L}_1, \qquad \mathcal{L}_1 = \Psi^\dagger\,(i\hbar\,\partial_t - H)\,\Psi + \hbar c\left(\zeta_+^*\,\nabla\cdot\boldsymbol{\psi}_+ + \zeta_-^*\,\nabla\cdot\boldsymbol{\psi}_- + \text{c.c.}\right), \tag{66}$$

where $\zeta_\pm$ are Lagrange multipliers for the constraint and c.c. denotes the complex conjugate. In the standard representation the first term is $\hbar c\,\bar\Psi_D\,i\beta^\mu\partial_\mu\Psi_D$. It is real up to a total derivative. Variation with respect to $\boldsymbol{\psi}^*_\pm$, at fixed $\boldsymbol{\psi}_\pm$, gives

$$i\,\partial_t\boldsymbol{\psi}_\pm = \pm c\,\nabla\times\boldsymbol{\psi}_\pm + c\,\nabla\zeta_\pm, \tag{67}$$

and variation with respect to $\zeta^*_\pm$ gives $\nabla\cdot\boldsymbol{\psi}_\pm = 0$. The divergence of the first equation is then $\nabla^2\zeta_\pm = 0$, whose only solution that vanishes at infinity is $\zeta_\pm = 0$, and the wave equation (22) with its constraint is recovered. The multipliers are needed only to state the constraint. The wave equation preserves it, since $\partial_t(\nabla\cdot\boldsymbol{\psi}_\pm) = 0$.

By the result of the preceding subsection $\mathcal{L}_1$ is not a Lorentz scalar. Without the multiplier terms it is $\mathcal{L}_1 = -i\hbar\left(\boldsymbol{\psi}^*_+\cdot\boldsymbol{\mathcal{J}}_+ + \boldsymbol{\psi}^*_-\cdot\boldsymbol{\mathcal{J}}_-\right)$, where $\boldsymbol{\mathcal{J}}_+$ is the spatial part of the four-vector (59)

built from $\boldsymbol{\psi}_+$, and $\boldsymbol{\mathcal{J}}_-$ the same with $\boldsymbol{\psi}_-$ and $c \to -c$. Under a boost, by Eqs. (60), $\mathcal{L}_1$ changes by terms proportional to the components $\mathcal{J}^\mu_\pm$. It therefore vanishes on the solutions in every frame, and the equations derived from it are covariant. The action is not invariant off the solutions, however, and Lorentz invariance cannot be used in Noether's theorem for it.

The symmetries that S_0 and S_1 do have give the conserved quantities of Sec. V. Both actions are invariant under the change of phase $\Psi \to e^{-i\alpha}\Psi$. For S_1 the Noether current, divided by $\hbar$, is

$$j^0 = \Psi^\dagger \Psi, \qquad \mathbf{j} = c\,\Psi^\dagger \rho_3 \boldsymbol{\Sigma}\,\Psi, \tag{68}$$

the energy density and the energy flux of Eq. (31). That a phase symmetry should yield the energy, which belongs to time translations, is explained by the weight. For the action S_0 the charge of the phase symmetry is the norm $\sum_\lambda \int \mathrm{d}^3k |f_\lambda|^2$, the number of photons, and the charge of time translations is $\langle H \rangle$. The local action S_1 carries one more power of $\hbar\omega_k$, so each of its charges is that of S_0 with one more power of the energy: the phase symmetry gives $\langle H \rangle$ and time translation gives $\langle H^2 \rangle$. All of these are conserved, because for a free particle every function of the momentum is. Invariance under space translations gives in the same way $\langle \mathbf{p} \rangle$ for S_0 and $\langle H\mathbf{p} \rangle$ for S_1, and the actions are also invariant under the independent change of phase of the two halves, $\Psi \to e^{-i\alpha\rho_3}\Psi$, whose charges are $\langle \Lambda \rangle$ and $\langle H\Lambda \rangle$. The identification of the Noether current (68) of the local action with the energy current is therefore correct, and it is tied to the energy normalization of Ψ.

C. Lorentz-invariant action

A Lorentz-invariant action must be built from the bilinear scalar $\mathbf{F} \cdot \mathbf{F}$, and it requires variables other than the wave function itself, because $\mathbf{F} \cdot \mathbf{F}$ contains no derivatives. The variables are the potentials. Let

$$\mathbf{F}[\phi, \mathbf{A}] = \frac{1}{\sqrt{2}} \left[\sqrt{\varepsilon_0}\,(-\partial_t \mathbf{A} - \nabla\phi) + \frac{i}{\sqrt{\mu_0}}\,\nabla \times \mathbf{A} \right], \tag{69}$$

with real ϕ and $\mathbf{A}$. By construction the imaginary part of the equations $\mathcal{J}^\mu = 0$ holds identically, $\nabla \cdot \mathbf{B} = 0$ and $\partial_t \mathbf{B} = -\nabla \times \mathbf{E}$. The real part follows from the action

$$S_M = \int \mathrm{d}t \int \mathrm{d}^3r\ \mathrm{Re}\,(\mathbf{F} \cdot \mathbf{F}) = \int \mathrm{d}t \int \mathrm{d}^3r\ \frac{1}{2} \left(\varepsilon_0\,\mathbf{E}^2 - \frac{\mathbf{B}^2}{\mu_0} \right), \tag{70}$$

which is the Maxwell action. Its Lagrangian density is a Lorentz scalar by the orthogonality of L, and it equals $\frac{1}{2}\bar{\chi}\chi$ in the notation of Eq. (50). Variation with respect to ϕ gives $\nabla \cdot \mathbf{E} = 0$ and variation with respect to $\mathbf{A}$ gives $\partial_t \mathbf{E} = c^2 \nabla \times \mathbf{B}$, so that the stationarity of S_M is equivalent to Eqs. (19).

The action S_M is invariant under the Poincaré group, and Noether's theorem for translations, after the usual symmetrization, gives the energy-momentum tensor. Written with the Riemann–Silberstein vector it is sesquilinear,

$$T^{00} = \mathbf{F}^* \cdot \mathbf{F}, \qquad T^{0i} = -i\,(\mathbf{F}^* \times \mathbf{F})_i\,, \qquad T^{ij} = \delta_{ij}\,\mathbf{F}^* \cdot \mathbf{F} - F_i^* F_j - F_j^* F_i, \tag{71}$$

symmetric and traceless, the $(1,1)$ tensor found above. For a field of one helicity, T^{00} and cT^{0i} coincide with the current (68). The two action principles thus agree on the conserved densities. In S_M they arise from translations and belong to a tensor; in S_1 they arise from the phase symmetry of the energy-normalized wave function.

VIII. PARITY, TIME REVERSAL AND DUALITY

Under space reflection $\mathbf{E}(\mathbf{r},t) \to -\mathbf{E}(-\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t) \to \mathbf{B}(-\mathbf{r},t)$, so that $\mathbf{F}(\mathbf{r},t) \to -\mathbf{F}^*(-\mathbf{r},t)$. In the standard representation this is $\chi(\mathbf{r},t) \to -\beta^0\chi(-\mathbf{r},t)$, and for the wave function

$$P: \qquad \Psi_D(\mathbf{r},t) \to -\beta^0\,\Psi_D(-\mathbf{r},t), \qquad \Psi(\mathbf{r},t) \to -\rho_1\,\Psi(-\mathbf{r},t). \tag{72}$$

The wave equation (52) is invariant because $\beta^0\beta^0\beta^0 = \beta^0$ and $\beta^0\beta^i\beta^0 = -\beta^i$, the second sign being compensated by $\nabla \to -\nabla$. In the chiral representation parity exchanges the two halves of Ψ, as it must, since reflection reverses the momentum and not the spin and so reverses the helicity. A wave function with one half only would describe a particle without a mirror image, as the Weyl equation does. The photon needs the reducible representation $(1,0)\oplus(0,1)$ because electromagnetism is invariant under reflections.

Under time reversal $\mathbf{E}(\mathbf{r},t) \to \mathbf{E}(\mathbf{r},-t)$ and $\mathbf{B}(\mathbf{r},t) \to -\mathbf{B}(\mathbf{r},-t)$, so that $\mathbf{F}(\mathbf{r},t) \to \mathbf{F}^*(\mathbf{r},-t)$ and, for the classical field, $\chi(\mathbf{r},t) \to \beta^0\chi(\mathbf{r},-t)$. On the wave function time reversal is antiunitary,

$$\mathcal{T}: \qquad \Psi(\mathbf{r},t) \to \Psi^*(\mathbf{r},-t), \tag{73}$$

and the complex conjugation together with $t \to -t$ keeps the frequency positive. Each half of Ψ goes into itself, since time reversal reverses both the momentum and the spin and leaves the helicity unchanged. The wave equation is invariant because $(\rho_3\mathbf{S}\cdot\mathbf{p})^* = \rho_3\mathbf{S}\cdot\mathbf{p}$, both $\mathbf{S}$ and $\mathbf{p} = -i\hbar\nabla$ being imaginary.

Maxwell's equations without sources are also invariant under the duality rotation

$$\sqrt{\varepsilon_0}\,\mathbf{E} \to \sqrt{\varepsilon_0}\,\mathbf{E}\cos\alpha + \frac{\mathbf{B}}{\sqrt{\mu_0}}\sin\alpha, \qquad \frac{\mathbf{B}}{\sqrt{\mu_0}} \to \frac{\mathbf{B}}{\sqrt{\mu_0}}\cos\alpha - \sqrt{\varepsilon_0}\,\mathbf{E}\sin\alpha, \tag{74}$$

of which the exchange $\sqrt{\varepsilon_0}\mathbf{E} \to \mathbf{B}/\sqrt{\mu_0}$, $\mathbf{B}/\sqrt{\mu_0} \to -\sqrt{\varepsilon_0}\mathbf{E}$ is the case $\alpha = \pi/2$. On the Riemann–Silberstein vector it is a change of phase,

$$\mathbf{F} \to e^{-i\alpha}\,\mathbf{F}, \qquad \Psi \to e^{-i\alpha\rho_3}\,\Psi, \qquad \Psi_D \to e^{-i\alpha\rho_1}\,\Psi_D. \tag{75}$$

This is the symmetry of the first-order action found in Sec. VII, generated by the helicity. The conservation of the helicity of the free field and its connection with duality are due to Calkin [13]. For the wave function the statement is that H commutes with ρ_3.

IX. THE PHOTON IN A MEDIUM

In a linear, isotropic medium without dispersion or absorption, with permittivity $\varepsilon(\mathbf{r})$ and permeability $\mu(\mathbf{r})$, the fields are related by $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ and obey $\partial_t\mathbf{D} = \nabla \times \mathbf{H}$, $\partial_t\mathbf{B} = -\nabla \times \mathbf{E}$, $\nabla \cdot \mathbf{D} = 0$ and $\nabla \cdot \mathbf{B} = 0$. A medium does not act on the photon as a potential. A term $V\Psi$ in the wave equation would reintroduce $\hbar$ through $V/\hbar$ and would give a dispersion relation $\hbar\omega = c\hbar k + V$, which is not the relation $\omega = ck/n$ of a medium of refractive index n. What the medium changes is the speed of propagation and the relative weight of the electric and magnetic fields in the wave function. The local speed and impedance are

$$v(\mathbf{r}) = \frac{1}{\sqrt{\varepsilon\mu}}, \qquad Z(\mathbf{r}) = \sqrt{\frac{\mu}{\varepsilon}}, \tag{76}$$

and the Riemann–Silberstein vectors are defined as in Ref. [5],

$$\mathbf{F}_\pm = \frac{1}{\sqrt{2}}\left(\frac{\mathbf{D}}{\sqrt{\varepsilon}} \pm i\,\frac{\mathbf{B}}{\sqrt{\mu}}\right), \tag{77}$$

so that $|\mathbf{F}_\pm|^2$ is again the energy density. Taking the time derivative and using Maxwell's equations,

$$i\,\partial_t\mathbf{F}_\pm = \pm\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{\mu}}\,\nabla \times \frac{\mathbf{D}}{\varepsilon} \pm \frac{i}{\sqrt{\varepsilon}}\,\nabla \times \frac{\mathbf{B}}{\mu}\right], \tag{78}$$

and expressing $\mathbf{D}/\varepsilon$ and $\mathbf{B}/\mu$ through $\mathbf{F}_+$ and $\mathbf{F}_-$, we find

$$i\,\partial_t\mathbf{F}_\pm = \pm\left[\sqrt{v}\,\nabla \times \big(\sqrt{v}\,\mathbf{F}_\pm\big) + \frac{v}{2}\,\nabla(\ln Z) \times \mathbf{F}_\mp\right]. \tag{79}$$

In a uniform medium this is Eq. (19) with c replaced by $v = c/n$. In an inhomogeneous medium a gradient of the speed acts within each helicity, and the operator $\sqrt{v}\,\nabla \times \sqrt{v}$ is Hermitian, while a gradient of the impedance couples the two helicities. Helicity is conserved in any medium of uniform impedance, whatever its refractive index, and it changes on reflection at a surface where

the impedance is discontinuous. Equation (79) agrees with the equations of Ref. [5] for the photon in a medium. In six-component form it reads

$$i\,\partial_t \Psi = \left[\rho_3\,\sqrt{v}\,(\mathbf{S}\cdot\tfrac{1}{i}\nabla)\sqrt{v} + \frac{v}{2}\,\rho_2\,\mathbf{S}\cdot\nabla(\ln Z)\right]\Psi, \tag{80}$$

where we used $\mathbf{a}\times\mathbf{f} = -i(\mathbf{S}\cdot\mathbf{a})\mathbf{f}$ and $\rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Both terms in brackets are Hermitian, the second because ρ_2, $\mathbf{S}\cdot\nabla(\ln Z)$ and v are Hermitian and commute.

X. RELATION TO THE QUANTIZED FIELD

The construction of Sec. II was carried out for one particle. We now show that it is the one-photon sector of the quantized electromagnetic field. The field operators are expanded in the modes of positive frequency (29), with the amplitudes replaced by annihilation operators,

$$\hat{\mathbf{E}}(\mathbf{r},t) = \sum_\lambda \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\,\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0}}\,\left[\mathbf{e}_\lambda(\mathbf{k})\,\hat{a}_\lambda(\mathbf{k})\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + \text{h.c.}\right], \tag{81}$$

where h.c. is the Hermitian conjugate and

$$[\hat{a}_\lambda(\mathbf{k}), \hat{a}^\dagger_{\lambda'}(\mathbf{k}')] = \delta_{\lambda\lambda'}\,\delta^3(\mathbf{k}-\mathbf{k}'). \tag{82}$$

The operator $\hat{\mathbf{B}}$ follows from $\partial_t\hat{\mathbf{B}} = -\nabla\times\hat{\mathbf{E}}$, which replaces $\mathbf{e}_\lambda$ by $\hat{\mathbf{k}}\times\mathbf{e}_\lambda/c = -i\lambda\,\mathbf{e}_\lambda/c$ in each mode. In the combination (17) the annihilation operator of helicity λ acquires the factor $(1+\lambda)/2$ and the creation operator the factor $(1-\lambda)/2$, so that

$$\hat{\mathbf{F}}(\mathbf{r},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\,\sqrt{\hbar\omega_k}\,\mathbf{e}_+(\mathbf{k})\,\left[\hat{a}_+(\mathbf{k})\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + \hat{a}^\dagger_-(\mathbf{k})\,e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right], \tag{83}$$

the classical solution (20) with operators for amplitudes. The operator $\hat{\mathbf{F}}$ annihilates photons of helicity $+1$ and creates photons of helicity -1. The Hamiltonian of the field is

$$\hat{H} = \int \mathrm{d}^3r\,{:}\hat{\mathbf{F}}^\dagger\cdot\hat{\mathbf{F}}{:} = \sum_\lambda \int \mathrm{d}^3k\,\hbar\omega_k\,\hat{a}^\dagger_\lambda(\mathbf{k})\hat{a}_\lambda(\mathbf{k}), \tag{84}$$

with the colons denoting normal ordering.

A general one-photon state is

$$|f\rangle = \sum_\lambda \int \mathrm{d}^3k\,f_\lambda(\mathbf{k})\,\hat{a}^\dagger_\lambda(\mathbf{k})|0\rangle, \tag{85}$$

with $|0\rangle$ the vacuum and the normalization (10). The matrix elements of the field between the vacuum and this state are

$$\langle 0|\hat{\mathbf{F}}(\mathbf{r},t)|f\rangle = \boldsymbol{\psi}_+(\mathbf{r},t), \qquad \langle 0|\hat{\mathbf{F}}^\dagger(\mathbf{r},t)|f\rangle = \boldsymbol{\psi}_-(\mathbf{r},t), \tag{86}$$

with $\boldsymbol{\psi}_\pm$ given by Eq. (15). The photon wave function is the amplitude for the field to annihilate the photon at the point $\mathbf{r}$ [5], and the weight $\sqrt{\hbar\omega_k}$, which was fixed in Sec. II by the energy, is the normalization of the field operator. Conversely, the field operator is obtained from the wave equation by second quantization: with a complete orthonormal set of one-photon states $|f_n\rangle$ and their wave functions $\Psi_n = (\boldsymbol{\psi}_{n+}, \boldsymbol{\psi}_{n-})^T$,

$$\hat{\mathbf{F}}(\mathbf{r},t) = \sum_n \left[\boldsymbol{\psi}_{n+}(\mathbf{r},t)\,\hat{c}_n + \boldsymbol{\psi}^*_{n-}(\mathbf{r},t)\,\hat{c}^\dagger_n\right], \qquad [\hat{c}_n, \hat{c}^\dagger_m] = \delta_{nm}, \tag{87}$$

where $\hat{c}^\dagger_n$ creates a photon in the state $|f_n\rangle$. Orthonormality refers to the scalar product (37), and this is the role of the nonlocal kernel in the quantized theory.

XI. CONCLUSIONS

The dispersion relation $E = c|\mathbf{p}|$ and the spin-one matrices lead, by Dirac's argument, to the first-order equation $i\partial_t\boldsymbol{\psi} = \pm c\nabla\times\boldsymbol{\psi}$ with $\nabla\cdot\boldsymbol{\psi} = 0$, which is the set of Maxwell equations for the Riemann–Silberstein vector. The requirement that the wave function describe both helicities with positive energy gives it six components and the transformation law $(1,0)\oplus(0,1)$, and we have given its finite Lorentz transformations in closed form and shown that the wave equation and the transversality condition together form a four-vector equation.

The analogy with the Dirac equation holds for the Hamiltonian, the spin and the standard and chiral representations, and it stops at the action. Because $(0,1)\otimes(1,0) = (1,1)$, the sesquilinear densities of the photon wave function are the components of the energy-momentum tensor, and there is neither a conserved four-vector current nor a Lorentz-scalar first-order Lagrangian of the Dirac kind. The local positive density that the wave function does have is the energy density $\Psi^\dagger\Psi$, and the probability is given by the nonlocal scalar product. The first-order action exists, is invariant on the solutions, and yields the energy current as the Noether current of the phase symmetry of the energy-normalized wave function; the invariant action is the Maxwell action, the real part of $\mathbf{F}\cdot\mathbf{F}$ with $\mathbf{F}$ expressed through the potentials. In a medium the wave equation keeps its form with the local speed of light, and the two helicities are coupled by the gradient of the impedance alone.

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